Estimating Geometric Aspects of Relative Satellite Motion Using Angles-Only Measurements

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This paper investigates the observability of relative satellite dynamics when only relative angle measurements are available. It has been shown previously that, for non-maneuvering motion, the Cartesian states are unobservable with angles-only measurements and, furthermore, none of the Cartesian states can be determined from these measurements. In this paper it will be shown that if the relative state is characterized instead using a geometric parameter set known as relative orbit elements, many of these parameters can be calculated directly from the angle measurements. This method allows one to quickly get an idea of the geometry of a relative trajectory based only on angle measurements.

I. Introduction

The primary objective of a satellite navigation system is to accurately determine a satellite's inertial position and velocity in space through onboard measurements pertaining to the satellite's position with respect to either the Earth or navigation beacons. Similarly, a *relative* navigation system allows the determination of the relative position and velocity of one satellite (a "deputy") with respect to a reference satellite (a "chief"). This type of system uses measurements pertaining to the deputy's position with respect to the chief. The design of autonomous relative navigation systems has been a popular research area the past few years. Such systems allow a spacecraft that is following a desired relative trajectory with respect to the chief to detect deviations from the desired trajectory, so it can then compute the required corrective action.

The most common measurement types for relative navigation systems, as with inertial navigation systems, are range and angles; in this case, the range is the distance from the deputy to the chief, and the angles are the relative azimuth and elevation indicating the direction from deputy to chief, with respect to a nominal plane (usually the chief's orbit plane). However, in some cases the deputy may not possess a ranging sensor, but can only measure relative angles. These angles (azimuth and elevation) can be immediately converted into a line-of-sight (LOS) unit vector from the deputy to the chief in the deputy's body-fixed frame. Then, using the satellite's attitude determination system and a series of coordinate rotations, this LOS vector can be expressed in whatever coordinate frame is desired.

It has been shown previously[1] that, for a relative navigation filter based on the Hill's-Clohessy-Wiltshire (HCW) equations[2-3], the Cartesian states characterizing the deputy's relative motion (i.e., its relative position and velocity components with respect to the chief) are unobservable with angles-only measurements, if neither satellite is maneuvering; namely, none of the Cartesian states can be recovered from these measurements. However, Ref. 1 indicated that the angle measurements do allow one to determine what "family" the relative trajectory belongs to. That is, it can be determined whether the trajectory is stationary with respect to the chief or drifting, as well as information about the shape of the trajectory. Thus, there ought to be some amount of geometric information recoverable from angle measurements. In this paper, the authors choose to characterize the relative state using a geometric parameter set known as relative orbit elements.[4] These are based on the solution to the HCW equations and give a geometric rather than a Cartesian representation of the relative state. It will be shown that, for a given trajectory, the values of several of the relative orbit elements (ROEs) can be calculated directly from the line-of-sight vectors. Because the angle measurements are imperfect, the calculated values will only be considered an

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estimate. Although not a full-blown relative navigation scheme, this method allows one to quickly get an idea of the geometry of a relative trajectory based only on angle measurements.

II. Relative Satellite Dynamics

Three major assumptions inherent in the HCW equations are that the only force modeled is that of a point mass gravitational field; the chief is in a circular orbit; and the distance between the satellites is small compared to their orbital radius. These assumptions yield the following linear time-invariant differential equations:

$$
\begin{aligned}
&\ddot{x} - 2n\,\dot{y} - 3n^2 x = 0 \\
&\ddot{y} + 2n\,\dot{x} = 0 \\
&\ddot{z} + n^2 z = 0
\end{aligned}
\tag{1}
$$

The equations are written in the local-vertical, local-horizontal (LVLH) coordinate frame, whose origin is at the chief satellite. In these equations, x is the component of the deputy's position vector relative to the chief in the radial direction positive away from the Earth; y is the along-track component positive along the velocity vector of the chief; z is the cross-track component perpendicular to the orbital plane of the chief; and n is the mean motion of the chief.

The solution to Equations (1) is:

$$
\begin{aligned}
x &= \frac{\dot{x}_0}{n}\sin(nt) - (3x_0 + \frac{2\dot{y}_0}{n})\cos(nt) + (4x_0 + \frac{2\dot{y}_0}{n}) \\
y &= \frac{2\dot{x}_0}{n}\cos(nt) + (6x_0 + \frac{4\dot{y}_0}{n})\sin(nt) - (6n\,x_0 + 3\dot{y}_0)t - \frac{2\dot{x}_0}{n} + y_0 \\
z &= \frac{\dot{z}_0}{n}\sin(nt) + z_0\cos(nt) \\
\dot{x} &= \dot{x}_0\cos(nt) + (3nx_0 + 2\dot{y}_0)\sin(nt) \\
\dot{y} &= -2\dot{x}_0\sin(nt) + (6nx_0 + 4\dot{y}_0)\cos(nt) - (6nx_0 + 3\dot{y}_0) \\
\dot{z} &= \dot{z}_0\cos(nt) - nz_0\sin(nt)
\end{aligned}
\tag{2}
$$

where x_0, y_0, etc, are conditions at some epoch time t_0, and t is the time since t_0. Consider the following change of coordinates from $x, y, z, \dot{x}, \dot{y}, \dot{z}$:[1-2]

$$
\begin{aligned}
a_e &= 2\sqrt{\left(\frac{\dot{x}}{n}\right)^2 + \left(3x + 2\frac{\dot{y}}{n}\right)^2} & x_d &= 4x + 2\frac{\dot{y}}{n} \\
y_d &= y - 2\frac{\dot{x}}{n} & \beta &= \operatorname{atan2}(\dot{x}, 3nx + 2\dot{y}) \\
z_{\max} &= \sqrt{\left(\frac{\dot{z}}{n}\right)^2 + z^2} & \gamma &= \operatorname{atan}(nz, \dot{z}) - \operatorname{atan2}(\dot{x}, 3nx + 2\dot{y})
\end{aligned}
\tag{3}
$$

where a_e, x_d, y_d, β, z_{max}, and γ are the ROEs. The inverse of this transformation is

$$
\begin{aligned}
x &= \frac{-a_e}{2}\cos\beta + x_d & \dot{x} &= \frac{a_e}{2} n \sin\beta \\
y &= a_e \sin\beta + y_d & \dot{y} &= a_e n \cos\beta - \frac{3}{2} n x_d \\
z &= z_{\max} \sin(\gamma + \beta) & \dot{z} &= z_{\max} n \cos(\gamma + \beta)
\end{aligned}
\tag{4}
$$

It was shown in Ref. 2 how the ROEs evolve with time:

$$
\begin{aligned}
a_e &= a_{e0} & x_d &= x_{d0} \\
y_d &= y_{d0} - \frac{3}{2} n x_{d0} t = y_{d0} - \frac{3}{2} n x_d t & \beta &= \beta_0 + nt \\
z_{\max} &= z_{\max 0} & \gamma &= \gamma_0 + nt
\end{aligned}
\tag{5}
$$

These equations are analogous to Eqns (2) for $x, y, z, \dot{x}, \dot{y}, \dot{z}$ in that they express the ROE values at any given time as a function of their initial (epoch) values and the time since epoch.

The parameterization of Eqns (4) reveals that the relative motion in the x-y plane of the deputy with respect to the chief is a superposition of periodic motion in x and y, with period equal to that of the chief 's orbit, and secular motion in y. Essentially, this is an elliptical path that is drifting in the *y*-direction at a rate of $-\frac{3}{2} n x_d$. The instantaneous center of the ellipse is (x_d, y_d). It has a semimajor axis of length a_e in the along-track direction and semiminor axis of length $a_e/2$ in the radial direction. β is a parametric angle (i.e. phase angle) indicating the location of the deputy satellite in its trajectory, with $\beta = 0$ corresponding to the perigee location (the "bottom" of the ellipse). The relative motion, if the elliptical path were "frozen" at a point in time, is depicted in Figure 1. Although the ellipse is actually drifting, it has been frozen in order to conveniently label the ROEs. The z motion, according to the HCW model, is purely sinusoidal and independent of x and y.

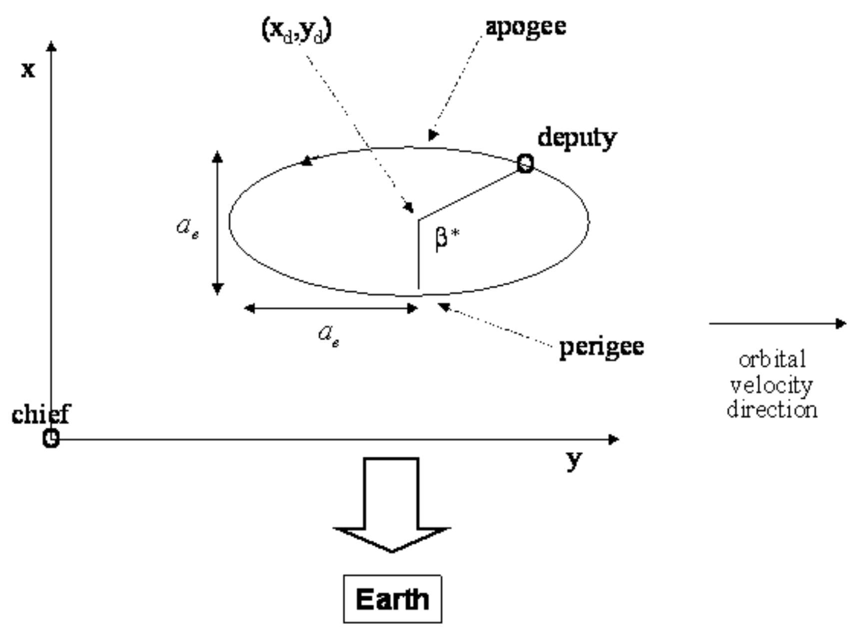

Figure 1. Planar Projection of Relative Motion Trajectory with Relative Orbit Elements Labeled.

III. Relationship Between LOS Vectors and ROEs

For this exercise it will be assumed that Line of Sight (LOS) unit vector has already been obtained and has the components i_x, i_y, and i_z, pointing along the LVLH x, y and z axes respectively. The LOS vector can be expressed as

$$LOS = \frac{\bar{r}}{r} \tag{6}$$

where $\bar{r}$ is the relative position vector between the two satellites. Since

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} \tag{7}$$

we can rewrite the LOS vector in terms of the Cartesian relative states:

$$LOS = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \tag{8}$$

Substituting in Eqns (4) will yield an expression for the LOS vector in terms of ROEs:

$$\begin{bmatrix} i_{x0} \\ i_{y0} \\ i_{z0} \end{bmatrix} = \frac{\left[x_d - \frac{a_e}{2}cos(\beta_0)\right]\hat{i} + \left[y_{d0} + a_e sin(\beta_0)\right]\hat{j} + \left[z_{max} sin(\psi_0)\right]\hat{k}}{\sqrt{a_e^2\left(1 - \frac{3}{4}cos^2(\beta_0)\right) + a_e\left(2y_{d0} sin(\beta_0) - x_d cos(\beta_0)\right) + x_d^2 + y_{d0}^2 + z_{max}^2 sin^2(\psi_0)}} \tag{9}$$

$$\begin{bmatrix} i_{x1} \\ i_{y1} \\ i_{z1} \end{bmatrix} = \frac{\left[x_d - \frac{a_e}{2}cos(\beta_1)\right]\hat{i} + \left[y_{d1} + a_e sin(\beta_1)\right]\hat{j} + \left[z_{max} sin(\psi_1)\right]\hat{k}}{\sqrt{a_e^2\left(1 - \frac{3}{4}cos^2(\beta_1)\right) + a_e\left(2y_{d1} sin(\beta_1) - x_d cos(\beta_1)\right) + x_d^2 + y_{d1}^2 + z_{max}^2 sin^2(\psi_1)}} \tag{10}$$

Thus, if we acquire the LOS vector (i.e. angle measurements) at several times along a trajectory, eventually we will have enough equations to solve for the unknowns (ROEs).

IV. Results

Equations (9) and (10) represent six relationships and six unknowns. Even though these equations are extremely nonlinear, and some of the ROE's are not observable, we are postulating that some of the ROE values can be found either explicitly or numerically.

A. Families of Solutions given two LOS Vectors

Having a known β_0 and ψ_0, greatly simplifies the problem of solving for the other LOS values. β and ψ are periodic values that could be obtained in a number of ways. One simple way is to derive β and ψ from the azimuth and elevation angle histories. $\beta = 0$ when the elevation angle is at a local minimum. $\psi = 0$ when the azimuth angle ascends through zero. An example using the initial conditions below may be found in Figure 2.

$$a_e = .03\,km$$
$$x_d = .01\,km$$
$$y_{d0} = .6\,km$$
$$\beta_0 = 0\pi/180\,radians$$
$$z_{max} = 0.01\,km$$
$$\psi_0 = 0\pi/180\,radians$$

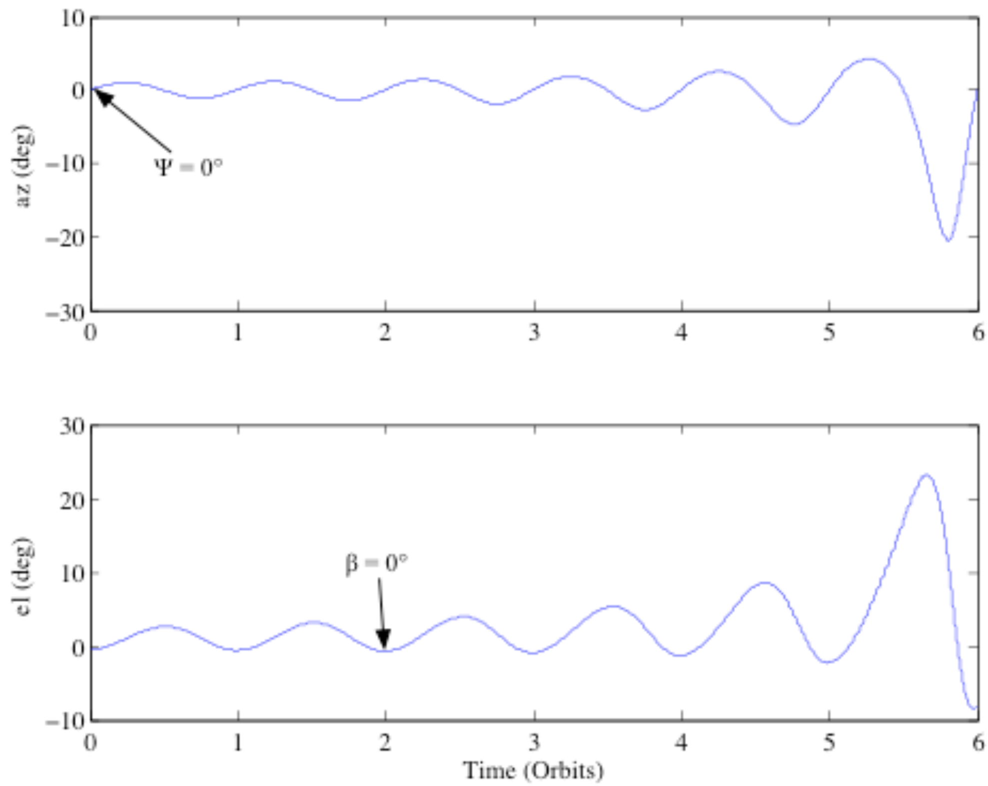

Figure 2. Example of solving for ψ and β from Azimuth and Elevation histories.

Equations (9) and (10), relating the LOS measurements to the ROE's, may be reformulated in terms of the range between the chief and deputy.

$$\hat{i}_{r0} = \frac{\left[x_d - \frac{a_e}{2}cos(\beta_0)\right]\hat{i} + \left[y_{d0} + a_e sin(\beta_0)\right]\hat{j} + \left[z_{max} sin(\psi_0)\right]\hat{k}}{R_0}$$
$$\hat{i}_{r1} = \frac{\left[x_d - \frac{a_e}{2}cos(\beta_1)\right]\hat{i} + \left[y_{d1} + a_e sin(\beta_1)\right]\hat{j} + \left[z_{max} sin(\psi_1)\right]\hat{k}}{R_1} \tag{11-12}$$

where

$$R_0 = \sqrt{a_e^2\left(1 - \frac{3}{4}cos^2(\beta_0)\right) + a_e\left(2y_{d0}sin(\beta_0) - x_d cos(\beta_0)\right) + x_d^2 + y_{d0}^2 + z_{max}^2 sin^2(\psi_0)} \tag{13}$$

and similarly for R_1. As a result, the measurements i_{r0} and i_{r1} correspond to a single solution associated with an assumed value for R_0. The result is a whole family of solutions dependent on the value of R_0.

Once an assumed value for R_0 is selected, z_{max} may be solved by noting that

$$z_{max} = \frac{R_0 i_{z0}}{sin(\psi_0)} \tag{14}$$

.

Rearranging Equations (11) and (12) results in the equation below:

$$a_e = \frac{2(x_d - R_0 i_{x0})}{cos(\beta_0)} = \frac{(R_0 i_{y0} - y_d)}{sin(\beta_0)} = \frac{2(x_d - R_1 i_{x1})}{cos(\beta_1)} = \frac{(R_1 i_{y1} - y_{d1})}{sin(\beta_1)} \tag{15}$$

Solving for this family of solutions given i_{r0}, i_{r1}, Δt, β_0, and ψ_0 is necessarily a numerical process. Note that only the first three equations are necessary to constrain the problem, and the solution is represented by the intersection of three planes as seen in Figure 3. As the assumed value for R_0 is increase the magnitude of x_d, y_d, and a_e will increase.

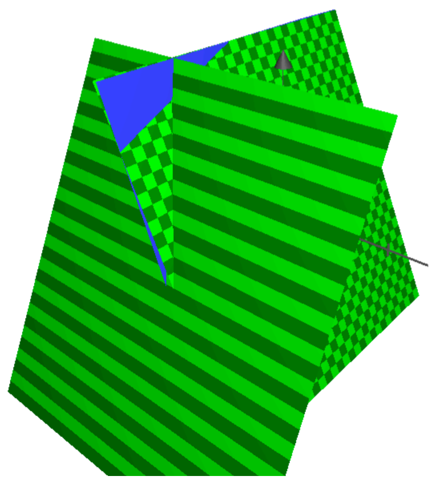

Figure 3. Representation of the solution for x_d, y_d, and a_e given an assumed initial range, two LOS vectors and the time they were apart.

An example solution set with unknown x_d, y_d, a_e, and z_{max} and the actual values below is shown in Figure 4.

$$\begin{aligned} a_e &= .01\,km \\ x_d &= .002\,km \\ y_{d0} &= .03\,km \\ \beta_0 &= 80\pi/180\,radians \\ z_{max} &= 0.01\,km \\ \psi_0 &= 5\pi/180\,radians \end{aligned}$$

$$\begin{aligned} \omega &= 0.001085410283934\,\mathrm{rad/sec} \\ \Delta t &= 198\,\mathrm{sec} \\ \hat{i}_{r0} &= \begin{bmatrix} 0.028383618444956 \\ 0.999358093716378 \\ 0.021857967141048 \end{bmatrix} \\ \hat{i}_{r1} &= \begin{bmatrix} 0.055713307466721 \\ 0.995603147336654 \\ 0.075302060957631 \end{bmatrix} \end{aligned}$$

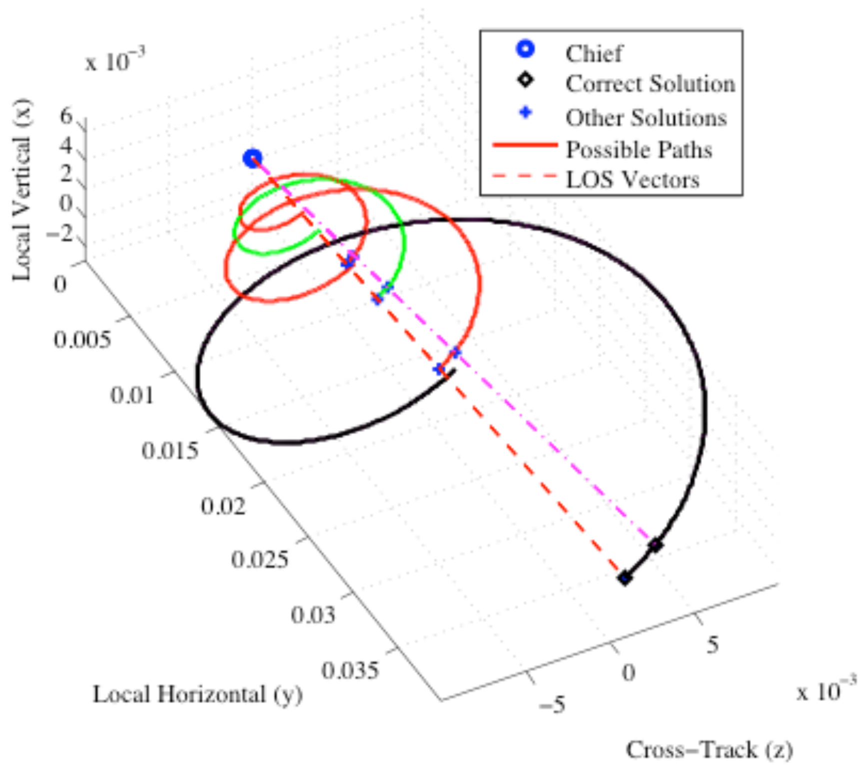

Figure 4. Representation of some of the possible trajectories given two LOS vectors, Δt, ω, β_0, and ψ_0.

B. Solutions Given Additional Info

In the case where a_e is known, it is a simple matter to numerically solve Equation (15) for x_d, y_{d0} , and z_{max} as long as a_e is nonzero. Numerical solver ambiguity about the sign of z_{max} can be eliminated by also solving

$$z_{max} = \frac{R_0 i_{z0}}{sin(\psi_0)} = \frac{R_1 i_{z1}}{sin(\psi_1)} \tag{16}$$

Similarly, in the case where x_d is known then it is a simple matter to numerically solve Equations (15) and (16) for a_e, y_{d0} , and z_{max} as long as x_d is nonzero. Note that the parts of Equation (15) that contain x_d must be used in order to obtain the correct solution. In the case where y_d is known, it is a simple matter to numerically solve Equations (15) and (16) for a_e, x_d, and z_{max} as long as y_d is nonzero. Note that the parts of Equation (15) that contain y_d must be used in order to obtain the correct solution. In the case where z_{max} is known, it is a simple matter to numerically solve Equations (15) and (16) for a_e, x_d , and y_d as long as z_{max} is nonzero.

C. Solutions for Special Cases

In the case where $a_e = 0$ than the following will be true:

$$i_{r0} = \begin{bmatrix} x_d \\ y_{d0} \\ z_{max} sin(\psi_0) \end{bmatrix} \frac{1}{R_0} \qquad i_{r1} = \begin{bmatrix} x_d \\ y_{d0} - \frac{3}{2}\omega x_d \Delta t \\ z_{max} sin(\psi_1) \end{bmatrix} \frac{1}{R_1}$$

$$R_1 = \sqrt{x_d^2 + y_{d1}^2 + z_{max}^2 s^2(\psi_1)}$$

$$R_0 = \text{Arbitrary range value} \tag{17}$$

If x_d=0 than the following holds true:

$$i_{r0} = \begin{bmatrix} -\frac{a_e}{2}c(\beta_0) \\ y_{d0} + a_e s(\beta_0) \\ z_{max} sin(\psi_0) \end{bmatrix} \frac{1}{R_0} \qquad i_{r1} = \begin{bmatrix} -\frac{a_e}{2}c(\beta_1) \\ y_{d0} + a_e s(\beta_1) \\ z_{max} sin(\psi_1) \end{bmatrix} \frac{1}{R_1}$$

$$R_1 = \sqrt{a_e^2\left(10\frac{3}{4}c^2\beta_1\right) + a_e 2y_{d1}s(\beta_1) + y_{d1}^2 + z_{max}^2 s^2(\psi_1)}$$

$$R_0 = \text{Arbitrary range value} \tag{18}$$

In a practical sense, this means that i_x and i_y and i_z will follow the same track every orbit. Two examples are seen in Figure 5. This will hold true even when β and ψ are unknown. Note that the y axis is a normalized value with respect to the magnitude of the unit vector.

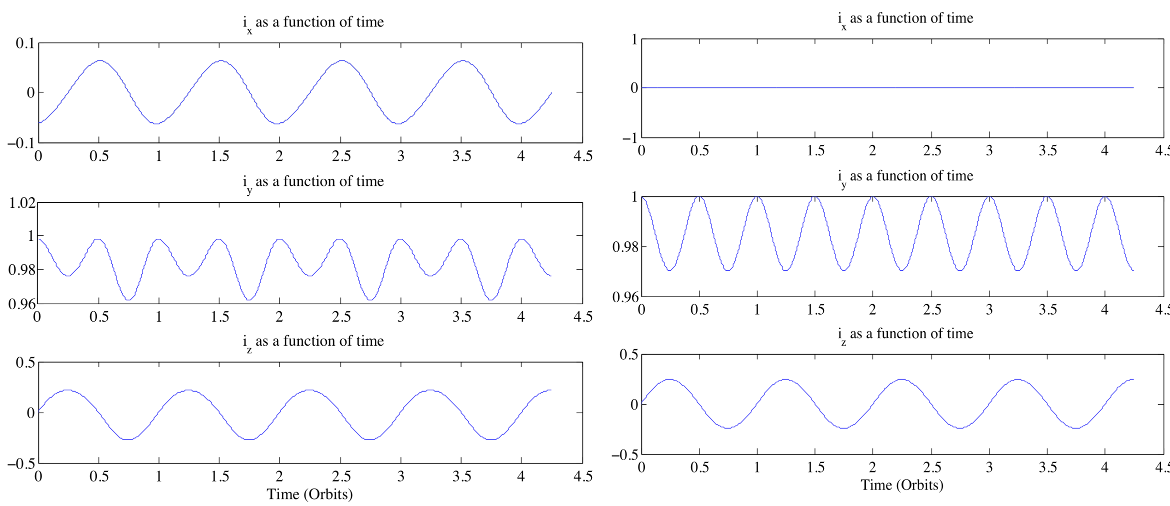

Figure 5. LOS measurement histories when x_d = 0.

Unless x_d =0, y_d will equal zero for only an instant. This is not very useful, so only the situation when both x_d and y_d=0 will be examined. In this case, the following will hold true:

$$i_{r0} = \begin{bmatrix} -\frac{a_e}{2}c(\beta_0) \\ a_e s(\beta_0) \\ z_{max} sin(\psi_0) \end{bmatrix} \frac{1}{R_0} i_{r1} = \begin{bmatrix} -\frac{a_e}{2}c(\beta_1) \\ a_e s(\beta_1) \\ z_{max} sin(\psi_1) \end{bmatrix} \frac{1}{R_1}$$

$$R_1 = \sqrt{a_e^2\left(10\frac{3}{4}c^2\beta_1\right) + z_{max}^2 s^2(\psi_1)}$$

$$R_0 = \text{Arbitrary range value} \tag{19}$$

As in the case where x_d =0, i_x and i_y and i_z will follow the same track every orbit. However, the i_y component will oscillate about 0 as seen in Figure 6. This will hold true even when β and ψ are unknown.

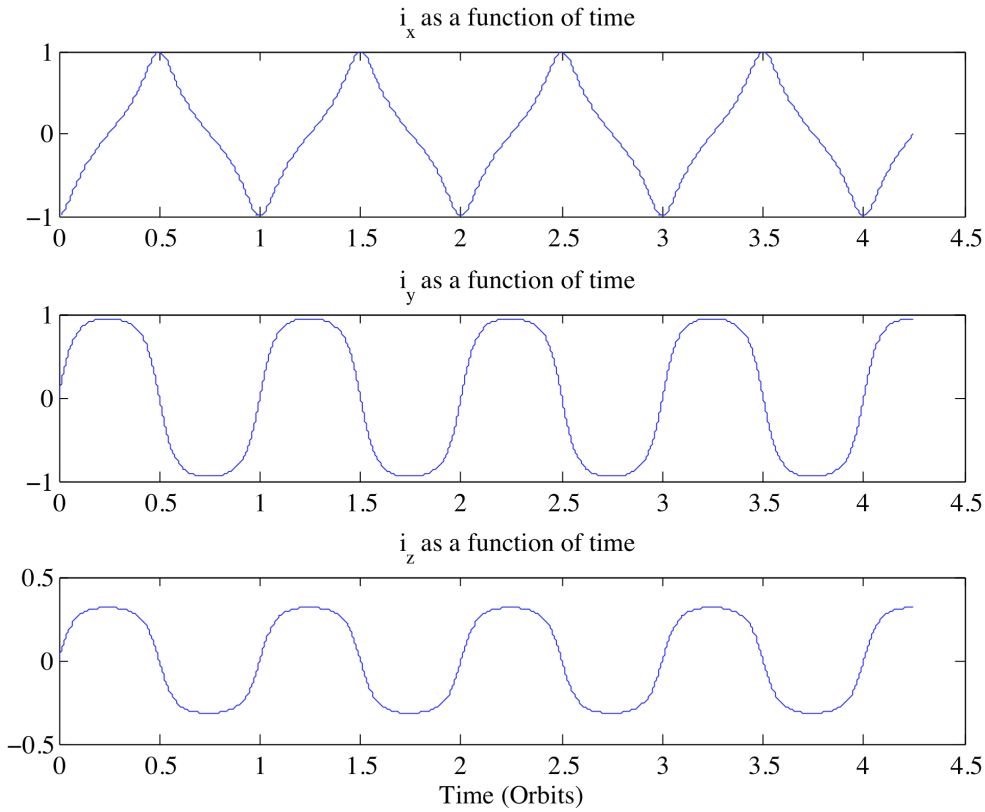

Figure 6. LOS measurement history as a function of time when x_d and y_d = 0.

If z_{max} =0 then the following holds:

$$i_{r0} = \begin{bmatrix} x_d - \frac{a_e}{2}c(\beta_0) \\ y_{d0} + a_e s(\beta_0) \\ 0 \end{bmatrix} \frac{1}{R_0} \qquad i_{r1} = \begin{bmatrix} x_d - \frac{a_e}{2}c(\beta_1) \\ y_{d1} + a_e s(\beta_1) \\ 0 \end{bmatrix} \frac{1}{R_1}$$

$$R_1 = \sqrt{a_e^2\left(10\frac{3}{4}c^2\beta_1\right) + a_e(2y_{d1}s(\beta_1) - x_d c(\beta_1)) + x_d^2 + y_{d1}^2}$$

$$R_0 = \text{Arbitrary range value} \tag{20}$$

Thus, if two LOS measurements, not taken a half period apart, both have an i_z component of 0, then z_{max} must equal zero. This will hold true even when ψ and β are unknown.

D. Analytical Solutions with a Single LOS Measurement

Squaring Eqn (9) and then multiplying both sides by the denominator (here, cos(β) is replaced by c(β), and sin(β) by s(β)), results in:

$$
\begin{bmatrix} i_x^2 \\ i_y^2 \\ i_z^2 \end{bmatrix} \left(a_e^2 \left(1 - \frac{3}{4}c^2(\beta)\right) + a_e\left(2y_d s(\beta) - x_d c(\beta)\right) + x_d^2 + y_d^2 + z_{max}^2 s^2(\psi) \right) = \begin{bmatrix} \frac{a_e^2}{4}c^2(\beta) - a_e x_d c(\beta) + x_d^2 \\ a_e^2 s^2(\beta) + 2a_e y_d s(\beta) + y_d^2 \\ z_{max}^2 s^2(\psi) \end{bmatrix} \tag{21}
$$

This is a form that allows analytical solutions for many of the ROE's.

In the special case where a_e is the only unknown and all measurements are perfect, it is possible to analytically solve for a_e given any component of a single LOS vector ($i_r = [i_x\ i_y\ i_z\ ]^T$), but using all three often eliminates multiple positive and real solutions. This is done as follows. Collecting like terms results in the following quadratic equations:

$$
\begin{aligned}
\left\{ i_x^2 \left(1 - \frac{3}{4}c^2(\beta)\right) - \frac{c^2(\beta)}{4} \right\} a_e^2 + \left\{ i_x^2(2y_d s(\beta)) + x_d c(\beta) \right\} a_e + \ldots \\
\left\{ i_x^2(x_d^2 + y_d^2 + z_{max}^2 s^2(\psi)) - x_d^2 \right\} = 0 \\
\left\{ i_y^2 \left(1 - \frac{3}{4}c^2(\beta)\right) - s^2(\beta) \right\} a_e^2 + \left\{ i_y^2(2y_d s(\beta)) - 2y_d s(\beta) \right\} a_e + \ldots \\
\left\{ i_y^2(x_d^2 + y_d^2 + z_{max}^2 s^2(\psi)) - y_d^2 \right\} = 0 \\
\left\{ i_z^2 \left(1 - \frac{3}{4}c^2(\beta)\right) \right\} a_e^2 + \left\{ i_z^2(2y_d s(\beta) + x_d c(\beta)) \right\} a_e + \ldots \\
\left\{ i_z^2(x_d^2 + y_d^2 + z_{max}^2 s^2(\psi)) - z_{max} s(\psi) \right\} = 0
\end{aligned} \tag{22)-(24}
$$

These equations can be solved by way of the quadratic equation. Examples show that they will share at least one common, real solution unless the coefficients go to zero (e.g. $x_d = 0$ and $\beta = 0$ or 180°, or when $y_d = x_d = 0$). When $i_x = 0$, then $a_e = 2x_d/c(\beta)$. If $i_y = 0$, then $a_e = -y_d/s(\beta)$. If $i_z = 0$ than no useful information results.

A more intuitive understanding of these equations may result when one assumes that the deputy and deputy or in the same orbit plane.(i.e. $z_{max} = 0$). For example, given the following ROE's and associated LOS vectors:

$$
\begin{aligned}
a_e &= unknown\ (but\ actually\ .04\,km) \\
x_d &= .01\,km \\
y_d &= -.01\,km \qquad\qquad i_x = -0.064502287668193 \\
\beta &= 55\pi/180\,radians \qquad\qquad i_y = 0.997917559162865
\end{aligned}
$$

Eqns (22) and (23) yield:

$$
-0.0791135a_e^2 + 0.0056437a_e - .000099 = 0
$$

$$
0.0791135a_e^2 - 0.0056437a_e + .000099 = 0
$$

Equation (24) has only zeros as coefficients. The allowable solutions for a_e are 0.0313372 or 0.04 km as shown in Figure 7.

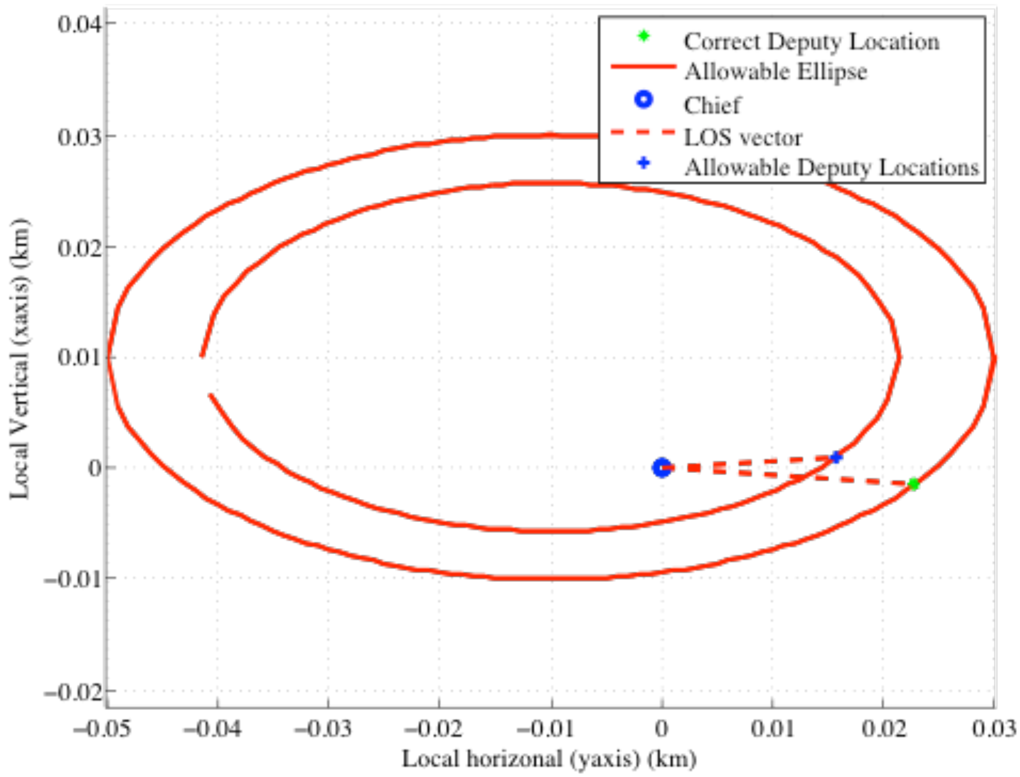

Figure 7. Possible solutions for ae given a 2D LOS vector: ae = 0.0313372 or 0.04 km.

The allowable solutions for a_e result from the fact that Eqns (22) and (23) are insensitive to the sign of the x and y components of the LOS vector in the 2D case. In the case where z_{max} is non-zero, the solutions for Eqns (22)-(24) may also result in multiple solutions, but they will be skewed from the results seen above based on the values of z_{max} and ψ. They will also share only one common solution, making the problem explicitly solvable.

In the case where x_d is the only unknown, the following quadratic equations may be derived from Eqn (21):

$$\begin{aligned}
&\left\{i_x^2-1\right\}x_d^2+\left\{(1-i_x^2)a_e c(\beta)\right\}x_d\ldots\\
&\qquad+\left\{i_x^2\left[a_e^2(1-\frac{3}{4}c^2(\beta))+2a_e y_d s(\beta)+y_d^2+z_{max}^2 s^2(\psi)\right]-\frac{a_e^2}{4}c^2(\beta)\right\}=0\\
&\left\{i_y^2\right\}x_d^2+\left\{-i_y^2 a_e c(\beta)\right\}x_d\ldots\\
&\qquad+\left\{i_y^2 a_e^2(1-\frac{3}{4}c^2(\beta))+(i_y^2-1)(2a_e y_d s(\beta)+y_d^2)+i_y^2 z_{max}^2 s^2(\psi)-a_e^2 s^2(\beta)\right\}=0\\
&\left\{i_z^2\right\}x_d^2+\left\{-i_z^2 a_e c(\beta)\right\}x_d\ldots\\
&\qquad+\left\{i_z^2\left((1-\frac{3}{4}c^2(\beta))a_e^2+2y_d s(\beta)a_e+y_d^2+z_{max}^2 s^2(\psi)\right)-z_{max}s(\psi)\right\}=0
\end{aligned}$$

(25)-(27)

Note that if i_x=0 then $x_d = a_e/2c(\beta)$. If i_y or i_z = 0 than no useful information about x_d can be derived. Unlike a_e, all three equations will yield the same quadratic even in the 3D case. An example solution to these equations, given the ROE's and LOS measurements below, is x_d = 0.00454428 or 0.01 km as shown in Figure 8.

$$z_{max} = 0.01\,km$$

$$a_e = .03\,km \qquad \psi = 20\pi/180\,radians$$

$$x_d = unknown\ (but\ actually\ .01\,km) \qquad i_x = 0.162202680419542$$

$$y_d = -.01\,km \qquad i_y = 0.965572711469072$$

$$\beta = 61\pi/180\,radians \qquad i_z = 0.203370669790363$$

Unlike the case with a_e, the solutions to these quadratic equations are insensitive to the sign of the x and y components of the LOS vector even in the 3D case. Other methods must be applied to refine the answer. The simplest method is to see which solution matches the sign on i_x. To solve explicitly, another measurement must be taken. This is explored in section E.

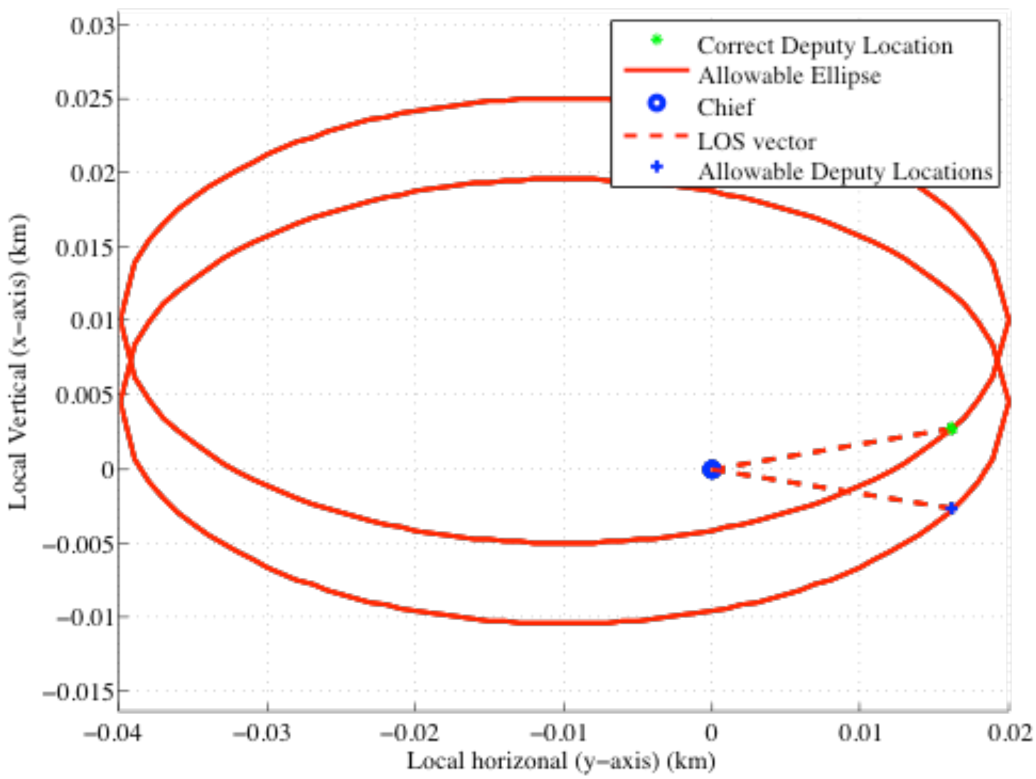

Figure 8. Possible solutions for xd given a 3D LOS vector xd=0.00454428 or 0.01km.

Solving Eq (21) for y_d yields the following quadratic equations:

$$\left\{i_x^2\right\} y_d^2 + \left\{2i_x^2 a_e s(\beta)\right\} y_d \ldots$$
$$+ \left\{(i_x^2-1)x_d^2 + (1-i_x^2)a_e c(\beta)x_d + i_x^2 a_e^2(1-\frac{3}{4}c^2(\beta)) + i_x^2 z_{max}^2 s^2(\psi) - \frac{a_e^2}{4}c^2(\beta)\right\} = 0$$
$$\left\{i_y^2-1\right\} y_d^2 + \left\{(i_y^2-1)2a_e s(\beta)\right\} y_d \ldots$$
$$+ \left\{i_y^2(x_d - a_e c(\beta))x_d + i_y^2 a_e^2(1-\frac{3}{4}c^2(\beta)) + i_y^2 z_{max}^2 s^2(\psi) - a_e^2 s^2(\beta)\right\} = 0$$
$$\left\{i_z^2\right\} y_d^2 + \left\{i_z^2 2a_e s(\beta)\right\} y_d \ldots$$
$$+ \left\{i_z^2\left(a_e^2(1-\frac{3}{4}c^2(\beta)) - a_e x_d c(\beta) + x_d^2 + z_{max}^2 s^2(\psi)\right) - z_{max}^2 s^2(\psi)\right\} = 0$$

(28)-(30)

Like x_d, these three equations will yield the same results even in the 3D case. An example solution to these equations, given the ROE's and LOS vector below, is y_d =-0.03697245 or 0.03 km as shown in Figure 9.

$$z_{max} = 0.01\,km$$
$$a_e = .04\,km \qquad \psi = 20\pi/180\,radians$$
$$x_d = .01\,km \qquad i_x = -0.282789522716489$$
$$y_d = unknown\ (but\ actually\ .03\,km) \qquad i_y = 0.954217664321781$$
$$\beta = 5\pi/180\,radians \qquad i_z = 0.097461453601231$$

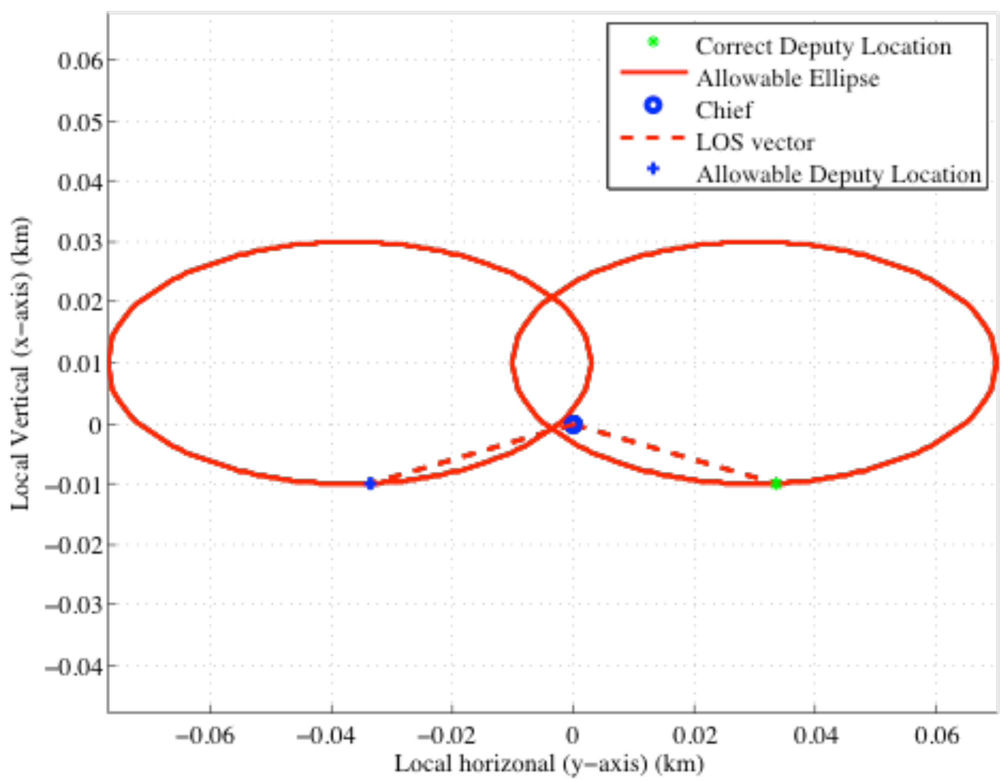

Figure 9. Possible solutions for yd given a 3D LOS vector: y_d =-0.03697245 or 0.03 km

Similarly to xd, the solution to these quadratic equations are insensitive to the sign of the x and y components of the

LOS vector even in the 3D case. Other methods must be applied to refine the answer. The simplest method is to see which solution matches the sign on iy. To solve explicitly, another measurement must be taken. This is done in section E.

Solving Eq (21) for z_{max} results in the following equations:

$$\left\{i_x^2 s^2(\psi)\right\} z_{max}^2 + \ldots$$
$$\left\{i_x^2\left(a_e^2(1-\frac{3}{4}c^2(\beta)) + a_e(2y_d s(\beta) - x_d c(\beta)) + x_d^2 + y_d^2\right) - \frac{a_e^2}{4}c^2(\beta) + a_e x_e c(\beta) - x_d^2\right\} = 0$$
$$\left\{i_y^2 s^2(\psi)\right\} z_{max}^2 + \ldots$$
$$\left\{i_z^2\left(a_e^2(1-\frac{3}{4}c^2(\beta)) + a_e(2y_d s(\beta) - x_d c(\beta)) + x_d^2 + y_d^2\right) - a_e^2 s^2(\beta) - 2a_e y_d s(\beta) - y_d^2\right\} = 0$$
$$\left\{(i_z^2 - 1)s(\psi)\right\} z_{max}^2 + \ldots$$
$$\left\{i_z^2\left(a_e^2(1-\frac{3}{4}c^2(\beta)) + a_e(2y_d s(\beta) - x_d c(\beta)) + x_d^2 + y_d^2\right)\right\} = 0$$

(31)-(33)

Solving using the quadratic equation results in solutions in the form: $\frac{\pm\sqrt{-4AC}}{2A}$. Thus it can be seen that analytically, the positive real result is the correct solution.

Solve for ψ using the following equation

$$\psi = \frac{cos^{-1}(1-2K)}{2}$$

where

$$K = \frac{-i_x^2\left[a_e^2(1-\frac{3}{4}c^2(\beta)) + a_e(2y_d s(\beta) - x_d c(\beta)) + x_d^2 + y_d^2\right] + \frac{a_e^2}{4}c^2(\beta) - a_e x_e c(\beta) + x_d^2}{i_x^2 z_{max}^2} \quad or$$
$$= \frac{-i_y^2\left[a_e^2(1-\frac{3}{4}c^2(\beta)) + a_e(2y_d s(\beta) - x_d c(\beta)) + x_d^2 + y_d^2\right] + a_e^2 s^2(\beta) + 2a_e y_d s(\beta) + y_d^2}{i_y^2 z_{max}^2} \quad or$$
$$= \frac{-i_z^2\left[a_e^2(1-\frac{3}{4}c^2(\beta)) + a_e(2y_d s(\beta) - x_d c(\beta)) + x_d^2 + y_d^2\right]}{(i_z^2-1) z_{max}^2}$$

(34)-(36)

These solutions will only work if i_x or i_y or $i_z \neq 0$ respectively, otherwise there will be a zero in the denominator of K. The solutions are also unable to distinguish between quadrants I and II, or III and IV because an identity using sin(ψ) was used to derive the solutions. Comparing the possible solutions to the actual LOS measurements is a simple way to resolve this problem.

E. Analytical Solutions with Two LOS Measurements

Given two measurements, x_d can be solved for without the multiple solution problem found with a single measurement. Solving equations 9 and 10 for x_d results in the quadratic equations below:

Using the ix component of the LOS vector:

$$\bullet \left\{i_{x0}^2 - 1\right\} x_d^2 + \left\{(1 - i_{x0}^2) a_e c(\beta_0)\right\} x_d \ldots$$
$$+ \left\{ i_{x0}^2 \left[a_e^2 (1 - \frac{3}{4} c^2(\beta_0)) + 2 a_e y_{d0} s(\beta_0) + y_{d0}^2 + z_{max}^2 s^2(\psi_0) \right] - \frac{a_e^2}{4} c^2(\beta_0) \right\} = 0$$
$$\bullet \left\{ i_{x1}^2 \left[1 + \frac{9}{4} \omega^2 \Delta t^2 \right] - 1 \right\} x_d^2 + \left\{(1 - i_{x1}^2) a_e c(\beta_1) - 3 i_{x1}^2 \omega \Delta t (a_e s(\beta_1) + y_{d0})\right\} x_d \ldots$$
$$+ \left\{ i_{x1}^2 \left[a_e^2 (1 - \frac{3}{4} c^2(\beta_1)) + 2 a_e y_{d0} s(\beta_1) + y_{d0}^2 + z_{max}^2 s^2(\psi_1) \right] - \frac{a_e^2}{4} c^2(\beta_1) \right\} = 0$$

(37)-(38)

Using the i_y component of the LOS vector:

$$\bullet \left\{i_{z0}^2\right\} x_d^2 + \left\{-i_{z0}^2 a_e c(\beta_0)\right\} x_d \ldots$$
$$+ \left\{ i_{z0}^2 \left((1 - \frac{3}{4} c^2(\beta_0)) a_e^2 + 2 y_{d0} s(\beta_0) a_e + y_{d0}^2 + z_{max}^2 s^2(\psi_0) \right) - z_{max}^2 s^2(\psi_0) \right\} = 0$$
$$\bullet \left\{ i_{z1}^2 \left[1 + \frac{9}{4} \omega^2 \Delta t^2 \right] \right\} x_d^2 + \left\{(1 - i_{z1}^2) a_e c(\beta_1) - 3 i_{z1}^2 \omega \Delta t (a_e s(\beta_1) + y_{d0})\right\} x_d \ldots$$
$$+ \left\{ i_{z1}^2 \left[a_e^2 (1 - \frac{3}{4} c^2(\beta_1)) + 2 a_e y_{d0} s(\beta_1) + y_{d0}^2 + z_{max}^2 s^2(\psi_1) \right] - z_{max}^2 s^2(\psi_1) \right\}$$

(39)-(40)

Using the i_z component of the LOS vector:

$$\bullet \left\{i_{y0}^2\right\} x_d^2 + \left\{-i_{y0}^2 a_e c(\beta_0)\right\} x_d \ldots$$
$$+ \left\{ i_{y0}^2 a_e^2 (1 - \frac{3}{4} c^2(\beta_0)) + (i_{y0}^2 - 1)(2 a_e y_{d0} s(\beta_0) + y_{d0}^2) + i_{y0}^2 z_{max}^2 s^2(\psi_0) - a_e^2 s^2(\beta_0) \right\} = 0$$
$$\bullet \left\{ i_{y1}^2 \left[1 + \frac{9}{4} \omega^2 \Delta t^2 \right] - \frac{9}{4} \omega^2 \Delta t^2 \right\} x_d^2 \ldots$$
$$+ \left\{ -i_{y1}^2 \left[a_e c(\beta_1) + 3 \omega \Delta t (y_{d0} + a_e s(\beta_1)) \right] + 3 \omega \Delta t (y_{d0} + a_e s(\beta_1)) \right\} x_d \ldots$$
$$+ \left\{ i_{y1}^2 \left[a_e^2 (1 - \frac{3}{4} c^2(\beta_1)) + 2 a_e y_{d0} s(\beta_1) + y_{d0}^2 + z^2 s^2(\psi_1) \right] - 2 a_e y_{d0} s(\beta_1) - y_{d0}^2 - a_e^2 s^2(\beta_1) \right\} = 0$$

(41)-(42)

Each pair of equations shares only the correct solution. This can be empirically understood from the fact that the alternate solution for the first equation results exists because the component LOS value has been squared. Only the correct solution will match the propagated y_{d0} in the second equation.

Given two measurements, y_d can be solved for without the multiple solution problem found with a single measurement. Solving equations 2 and 3 for yd results in the quadratic equations below:

Using the i_x component of the LOS vector:

- $\left\{i_{x0}^2\right\} y_{d0}^2 + \left\{2i_{x0}^2 a_e s(\beta_0)\right\} y_d \ldots$
  $+ \left\{(i_{x0}^2 - 1)x_d^2 + (1 - i_{x0}^2)a_e c(\beta_0)x_d + i_{x0}^2 a_e^2(1 - \frac{3}{4}c^2(\beta_0)) + i_{x0}^2 z_{max}^2 s^2(\psi_0) - \frac{a_e^2}{4}c^2(\beta_0)\right\} = 0$
- $\left\{i_{x1}^2\right\} y_{d0}^2 + \left\{i_{x1}^2\left[2a_e s(\beta_1) - 3\omega x_d \Delta t\right]\right\} y_{d0} \ldots$
  $+ \{i_{x1}^2 \left[\frac{9}{4}\omega^2 x_d^2 \Delta t^2 - 3a_e \omega x_d \Delta t s(\beta_1) + x_d^2 - a_e x_d c(\beta_1) + a_e^2\left(1 - \frac{3}{4}c^2(\beta_1)\right) + z_{max}^2 s^2(\psi_1)\right] \ldots$
  $- x_d^2 + a_e c(\beta_1)x_d - \frac{a_e^2}{4}c^2(\beta_1)\} = 0$

(43)-(44)

Using the i_y component of the LOS vector:

- $\left\{i_{y0}^2 - 1\right\} y_{d0}^2 + \left\{(i_{y0}^2 - 1)2a_e s(\beta_0)\right\} y_d \ldots$
  $+ \left\{i_{y0}^2\left[x_d^2 - a_e c(\beta_0)x_d + a_e^2(1 - \frac{3}{4}c^2(\beta_0)) + z_{max}^2 s^2(\psi_0)\right] - a_e^2 s^2(\beta_0)\right\} = 0$
- $\left\{i_{y1}^2 - 1\right\} y_{d0}^2 + \left\{(i_{y1}^2 - 1)\left[2a_e s(\beta_1) - 3\omega x_d \Delta t\right]\right\} y_{d0} \ldots$
  $+ \{(i_{y1}^2 - 1)\left[\frac{9}{4}\omega^2 x_d^2 \Delta t^2 - 3a_e \omega x_d \Delta t s(\beta_1)\right] + i_{y1}^2\left[x_d^2 - a_e c(\beta_1)x_d + a_e^2(1 - \frac{3}{4}c^2(\beta_1)) + z_{max}^2 s^2(\psi_1)\right] ..$
  $- a_e^2 s^2(\beta_1)\} = 0$

(45)-(46)

Using the i_z component of the LOS vector:

- $\left\{i_{z0}^2\right\} y_{d0}^2 + \left\{i_{z0}^2 2a_e s(\beta_0)\right\} y_{d0} \ldots$
  $+ \left\{i_{z0}^2\left(a_e^2(1 - \frac{3}{4}c^2(\beta_0)) - a_e x_d c(\beta_0) + x_d^2 + z_{max}^2 s^2(\psi_0)\right) - z_{max}^2 s^2(\psi_0)\right\} = 0$
- $\left\{i_{z1}^2\right\} y_{d0}^2 + \left\{i_{z1}\left[2a_e s(\beta_1) - 3\omega x_d \Delta t\right]\right\} y_{d0} \ldots$
  $+ \{i_{z1}^2\left[\frac{9}{4}\omega^2 x_d^2 \Delta t^2 - 3a_e \omega x_d \Delta t s(\beta_1) + x_d^2 - a_e x_d c(\beta_1) + a_e^2\left(1 - \frac{3}{4}c^2(\beta_1)\right) + z_{max}^2 s^2(\psi_1)\right] \ldots$
  $- z_{max}^2 s^2(\psi_1)\} = 0$

(47)-(48)

Each pair of equations shares only the correct solution. This can be empirically understood from the fact that the alternate solution for the first equation results exists because the component LOS value has been squared. Only the correct solution will match the propagated y_{d0} in the second equation.

V. Conclusion

As the need increases for accurate relative satellite navigation systems (and ones that can operate given a sparse amount of measurement data), the methods described in this paper could become quite useful. Given only relative angle measurements (i.e. line-of-sight vectors) between two satellites and a model based only on two-body dynamics, the unobservability problem is inevitable. However, this method allows one to gain some knowledge of the geometric aspects of the satellite's relative trajectory. Such an algorithm could be incorporated into on-board flight software, employed by mission operators on the ground, or simply used as a post-processing tool for mission data. Future efforts will involve expanding the algorithm into a filter technique that can estimate relative orbit elements given a sizeable amount of angle measurement data.

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